Bose-Einstein condensation

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B.Sc Part-III
Paper-VI

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Bose-Einstein condensation :-

For an ideal bose gas of N molecules in a volume V the most probable no. of particle with energy ϵ_i is given by

$$\bar{n}_i(\epsilon_i) = \frac{g_i}{e^{\alpha+\beta\epsilon_i}-1}$$

where $\beta = \frac{1}{KT}$ & $\alpha = -\frac{\mu}{KT}$, μ = chemical pot. of the system, g_i is the degeneracy in the i^{th} ~~state~~ level. The value of μ can be determine as the fn. of N & T by the relation

$$N = \sum_{i=1}^{\infty} \bar{n}_i = \sum_{i=1}^{\infty} \frac{g_i}{e^{(\epsilon_i-\mu)/KT}-1} \quad \text{--- (1)}$$

We must have $\bar{n}_i \geq 0$ because the no. of Particle can not be negative. Therefore for boson gas at all temp. greater than zero. For all ϵ_i $e^{(\epsilon_i-\mu)/KT} \geq 1$. At the ground state if we choose $\epsilon_i = 0$. Then $e^{-\mu/KT} \geq 1$. This suggest that μ is negative or equal to zero ($\mu \leq 0$). For photon $\mu = 0$.

If we replace the summation by integration we have to replace g_i by the density of states. Therefore,

$$g(\epsilon)\,d\epsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\epsilon^{1/2}\,d\epsilon \quad \text{--- (2)}$$

Therefore eqn. (1) becomes

$$N = \int_0^\infty \frac{2\pi V}{h^3}(2m)^{3/2}\frac{\epsilon^{1/2}}{\frac{1}{\eta_a}e^{\beta\epsilon}-1}\,d\epsilon$$

where $\eta_a = e^{\beta\mu} \leq 1$ known as absolute activity.

we put $\epsilon/KT = x$

$\therefore d\epsilon = KT\,dx$

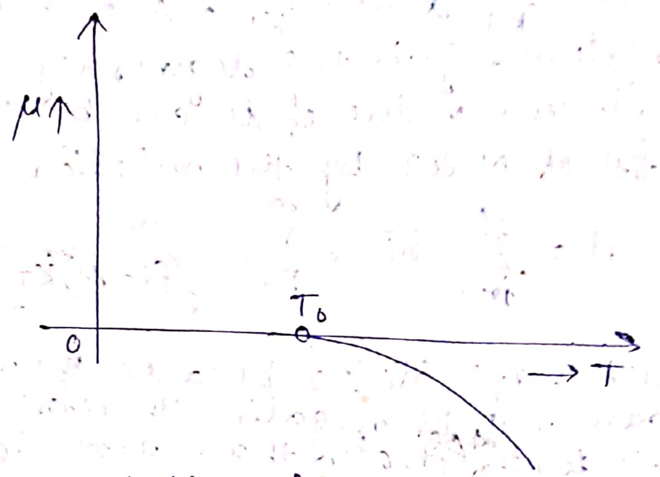
$$\therefore N = \frac{2\pi V}{h^3}(2m)^{3/2}(KT)^{3/2}\int_0^\infty \frac{x^{1/2}\,dx}{\frac{1}{\eta_a}e^{x}-1}$$

$$= \frac{V}{\lambda^3}F_{3/2}(\eta_a) \quad \text{--- (3)}$$

where $\lambda = \frac{h}{(2\pi mKT)^{1/2}}$ &

$$F_{3/2}(\eta_a) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}\, dx}{\frac{1}{\eta_a} e^{x} - 1}.$$

Eqn ③ has no solution for $T < T_0$. μ can not be positive nor ~~if~~ it become '-ve' again. The only possibility is for μ to make equal to zero.



The variation of μ vs T is shown in the fig. The difficulty occurs here due to the improper change of the sum into integral. At very low temp. this causes a serious error as large contribution is coming from the 1st few terms in eqn ①. For a boson gas there is no restriction on the no. of particle that can belong to a single state. Hence the large contribution comes from the 1st few terms in eqn ⑦ intact For $T \to 0$ The 1st term $\bar{n}_0$ approaches to the total no. of particle N. Because

$$\lim_{T \to 0} \bar{n}_0 = N = \frac{1}{e^{(\epsilon_0 - \mu)/KT} - 1}$$

$$\simeq \frac{KT}{\epsilon_0 - \mu} \quad [n_0 = \text{large}) \quad ④$$

Thus for very low temp. μ is very close to ϵ_0 so the population in the ground state is very large. This phenomenon is known as Bose-Einstein condensation. The reason behind the BE condensation is the behavior of chemical potential μ of the boson gas at low temp. It must be noted that the condensation refer to the condensation in momentum space and not in actual condensation in the gas.

For $T \to 0$ the sum $(N - \bar{n}_0)$ can be approximated by the integral without any serious errors.

$$\therefore N - \bar{n}_0 = \sum_1 \bar{n}_i$$

$$\simeq \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{\frac{1}{\eta_a} e^{\beta\epsilon} - 1}$$

$$= \frac{V}{\lambda^3} F_{3/2}(\eta_a) \quad \text{—— (5)}$$

Replacing V, we get,

$$N - \bar{n}_0 = N \left(\frac{T}{T_0}\right)^{3/2} \frac{F_{3/2}(\eta_a)}{2.612}$$

$$\therefore \bar{n}_0 = N \left[1 - \left(T/T_0\right)^{3/2} \frac{F_{3/2}(\eta_a)}{2.612} \right]$$

For low temp. μ is very close to zero and we can put $\eta_a = 1$.

$$\therefore \bar{n}_0 = N \left[1 - \left(T/T_0\right)^{3/2} \right] \qquad \boxed{\bar{n}_0 = N_0}$$

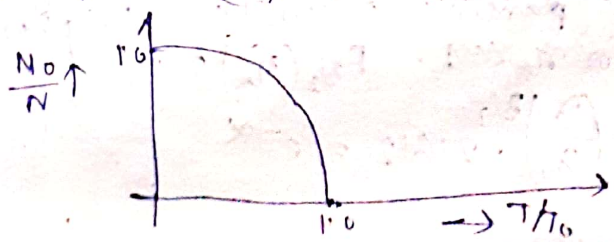
So the no. of particle in the exited state

$$N' = N - \bar{n}_0 = N \left(\frac{T}{T_0}\right)^{3/2}$$

At $T = T_0$, $\bar{n}_0 = 0$. at $T = 0$ $\bar{n}_0 = N$.

So as the temp. decreases below T_0, more and more particle begin to occupy ground state. The BE gas then degenerate and we are in the quantum region characterised by $\mu \to 0$. At temp $T = T_0$ is called degeneracy temp.

The plot (N_0/N) vs (T/T_0) is shown in the fig.



An alternative to defin T_0 we can define at critical volume V_0 such that at temp T

$$N = \frac{V_0}{\lambda^3} F_{3/2}(\eta_a = 1)$$

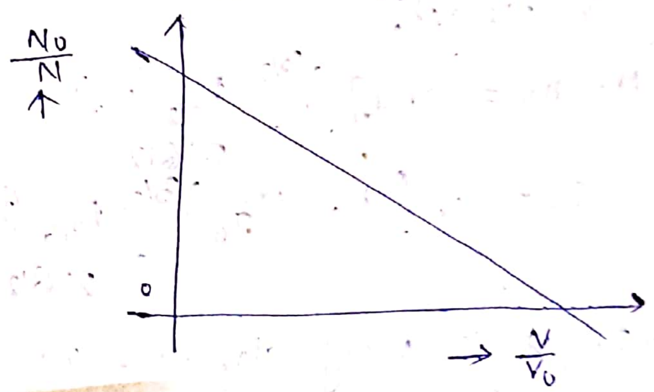
$$= V_0 \left(\frac{2\pi m K T}{h^2}\right)^{3/2} \times 2.612$$

$$\therefore N = N_0 + N \frac{V}{V_0} \cdot \frac{F_{3/2}(\eta_a)}{2.612}$$

$$\simeq N_0 + N \frac{V}{V_0}$$

$$\therefore N_0 = N\left(1 - \frac{V}{V_0}\right) \quad (V < V_0).$$

A plot of $\frac{N_0}{N}$ vs $\frac{V}{V_0}$ is shown in fig.



For $T > T_0$ we have $\eta_a < 1$. and the contribution of 1st term in eqn ①. [N_0 becomes negligible) and the 2nd term increases as $T^{3/2}$ when the bose gas is heated above T_0. Therefore

$$N = \frac{V}{\lambda^3} F_{3/2}(\eta_a)$$

$$\text{Now } F_{3/2}(\eta_a) = \left(\frac{T_0}{T}\right)^{3/2} F_{3/2}(\eta_a = 1)$$

$$\therefore N = \frac{V}{\lambda^3}\left(\frac{T_0}{T}\right)^{3/2} \times 2.612$$

For $T \gg T_0$ the ground state is practically empty and most of the particle occupy the higher excited state. For $\eta_a \ll 1$. $F_{3/2}(\eta_a) \simeq \eta_a$

$$\therefore \eta_a = \left(\frac{\lambda}{\lambda_0}\right)^{3/2} \cdot 2.612 = e^{-\alpha}$$

$\therefore \; \rho\lambda^3 = e^{-\alpha}$ (classical limit).

where $\rho = \dfrac{N}{V}$

Therefore the BE gas behaves as MB gas.

Energy of the bose-gas :-

The internal energy of the bose gas is given by

$$U = \sum_i \epsilon_i \bar{n}_i = \int_0^\infty \epsilon \, dn$$

$$= \int_0^\infty \frac{\epsilon \, g(\epsilon) \, d\epsilon}{e^{(\epsilon-\mu)/KT} - 1}$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2} \int_0^\infty \frac{\epsilon^{3/2} \, d\epsilon}{e^{(\epsilon-\mu)/KT} - 1}$$

$$= \frac{VKT}{\lambda^3} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty x^{3/2} \left(\frac{1}{\eta_a} e^{x} - 1\right)^{-1} dx$$

where $\epsilon/KT = x$.

and $\eta_a = e^{\beta\mu} \leq 1$.

$$= \frac{3}{2} KT \cdot \frac{V}{\lambda^3} f_{5/2}(\eta_a)$$

where $f_{5/2}(\eta_a) = \dfrac{1}{\frac{3}{4}\sqrt{\pi}} \displaystyle\int_0^\infty x^{3/2} \left(\frac{1}{\eta_a} e^{x} - 1\right)^{-1} dx$

$$= \frac{1}{\frac{3}{4}\sqrt{\pi}} \int_0^\infty \eta_a e^{-x} x^{3/2} \left(1 - \eta_a e^{-x}\right)^{-1} dx$$

$$= \frac{1}{\frac{3}{4}\sqrt{\pi}} \int_0^\infty \eta_a e^{-x} x^{3/2} \left(1 + \eta_a e^{-x} + \eta_a^2 e^{-2x} + \dots\right) dx$$

$$= \eta_a + \frac{\eta_a^2}{2^{5/2}} + \frac{\eta_a^3}{3^{5/2}} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{\eta_a^{\,n}}{n^{5/2}}$$

We shall consider two cases,

i) For $T < T_0$ (degenerate gas)

ii) For $T > T_0$ (nondegenerate gas).

For $T < T_0$, $\eta_a = 1$. and,

$$F_{5/2}(\eta_a = 1) = \zeta(5/2)$$

Therefore,

$$U_- = \frac{3}{2} KT \frac{V}{\lambda^3} \zeta(5/2)$$

we have, $N = \frac{V}{\lambda_0^3} \zeta(3/2)$

$$\therefore U_- = \frac{3}{2} NKT \left(T/T_0\right)^{3/2} \frac{\zeta(5/2)}{\zeta(3/2)}$$

$$= \frac{3}{2} NKT \left(T/T_0\right)^{3/2} \frac{1.341}{2.612}$$

$$= \frac{3}{2} NKT \left(T/T_0\right)^{3/2} 0.51.$$

From the above expression it is clear that the energy of the bose gas is less than that of the ideal MB gas.

For $T > T_0$,

$$U_+ = \frac{3}{2} KT \frac{V}{\lambda^3} F_{5/2}(\eta_a)$$

we have $N = \frac{V}{\lambda^3} F_{3/2}(\eta_a)$

$$\therefore U_+ = \frac{3}{2} NKT \frac{F_{5/2}(\eta_a)}{F_{3/2}(\eta_a)}$$

$$= \frac{3}{2} NKT \left[1 - 2^{-5/2} F_{3/2}(\eta_a) - 2\left(3^{-5/2} - 2^{-4}\right)\{F_{3/2}(\eta_a)\}^2 - \cdots\right].$$

$$\therefore U_+ = \frac{3}{2} NKT \left[1 - 0.462\left(\frac{T_0}{T}\right)^{3/2} - 0.023\left(\frac{T_0}{T}\right)^{3}\right].$$

This shows that the energy of bose gas still less than the Fermi gas.

Specific heat of Bose-gas :=>

we have,

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

we have two cases.

i) $T < T_0$

ii) $T > T_0$.

For $T < T_0$

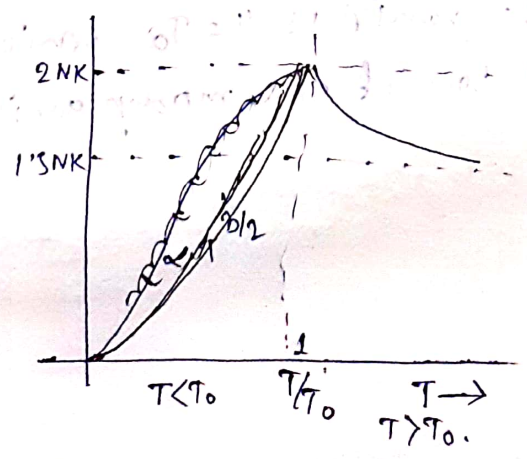
$$C_{V-} = \frac{15}{4} NK (T/T_0)^{3/2} \times 0.51.$$

$$= 1.926\, NK\, (T/T_0)^{3/2}$$

For $T > T_0$

$$C_{V+} = \frac{3}{2} NK \left[1 + 0.231 \left(\frac{T_0}{T}\right)^{3/2} + \cdots\cdots\right].$$

If we plot C_V Vs T



The sp. heat curve shows a kink which indicate that the bose consendasation is the 1st order phase transmission.

Entropy of Bose gas :=>

We have entropy

$$S = \int_0^T \frac{C_V\,dT}{T}$$

We have two cases.

i) $T < T_0$

ii) $T > T_0$

For $T < T_0$,

$$S_-(T) = \int_0^T \frac{C_{V-}\,dT}{T}$$

$$= \frac{2}{3} \times 1.926\,NK \left(T/T_0\right)^{3/2}$$

For $T > T_0$

$$S_+(T) = S(T_0) + \int_{T_0}^{T} \frac{C_{V+}\,dT}{T}$$

$$= S(T_0) + \frac{3}{2} NK\left[\ln\left(T/T_0\right) + \frac{2}{3} \times 0.231 \left(1 - T_0/T\right)^{3/2} + \cdots\right.$$

The entropy shows a certain drop for $T < T_0$ and it vanishes $T = T_0$ which is consisting 3rd low of thermodynamics.

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Pressure of ideal Bose gas.

We have,

$$P = \frac{2}{3}\frac{U}{V}$$

we have two cases; i) $T < T_0$
ii) $T > T_0$.

for $T < T_0$

$$P_- = \frac{2}{3}\frac{U_-}{V} = NK\left(\frac{T}{V_0}\right)$$

we have using $V/V_0 = (T/T_0)^{3/2}$

For $T > T_0$

$$P_+ = \frac{2}{3}\frac{U_+}{V} = \frac{NKT}{V}\left[1 - 0.462\frac{V_0}{V} - \cdots\right].$$

For condensation in a gas as we decrease the temp. The pressure also decreases.

Consider a gas composed of fixed no. of Boson (N) in a container of volume V. Show that the α of such a system always -ve & it is strictly increasing fn of T.

we have, $N = \int_0^\infty dn$

$$= \int_0^\infty \frac{g(\epsilon)\,d\epsilon}{e^{\alpha+\beta\epsilon}-1}$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{\epsilon^{1/2}\,d\epsilon}{e^{\alpha}e^{\epsilon/KT}-1}$$

If α is '-ve' then at the ground the the occupation no. is '-ve' which can not happen. α must non negative.

put $x = \epsilon/KT$

$\epsilon = KTx$

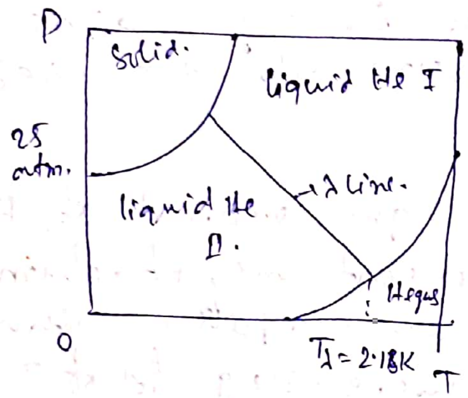
$d\epsilon = KT\,dx$.

$$= \frac{2\pi V}{h^3}(2mKT)^{3/2}\left[\int_0^\infty \frac{x^{1/2}\,dx}{e^{\alpha}e^{x}-1}\right].$$

As T increase the 1st term increases. So the value of the integral has to decrease, because the no. of particle is fixed. Therefore, α must increase as we increase the temp.

Liquid helium (λ transition) :->

An ^4He atom contain two proton and two neutron so it obey BE condensation. Thus liquid ^4He undergoes BE condensation. The phase diagram of ^4He is shown in fig.



As liq. ^4He in contact in vapour is cool it begins to show dramatic change in its property at $T = 2.18K$. For $T > T_\lambda$ it is a normal liq. called Helium-I. For $T < T_\lambda$ the liq. ^4He begin to show some remarkable property such as (i) zero viscosity and zero entropy. It is called super fluid. The transition between ^4He I – ^4He II is called λ transition. The λ-transition is the form of ~~Bord~~ Bose condensation.